

ON DERIVATIONS ANNIHILATING A MAXIMAL ABELIAN SUBALGEBRA

BY

GEOFFREY L. PRICE¹

ABSTRACT. Let $\mathcal{A}$ be an AF C^* -algebra, and let δ be a closed $*$ -derivation which annihilates the maximal abelian subalgebra $\mathcal{C}$ of diagonal elements of $\mathcal{A}$. Then we show that δ generates an approximately inner C^* -dynamics on $\mathcal{A}$, and that δ is a commutative $*$ -derivation. Any two closed $*$ -derivations vanishing on $\mathcal{C}$ are shown to be strongly commuting. More generally, if δ is a semiderivation on $\mathcal{A}$ which vanishes on $\mathcal{C}$, we prove that δ is a generator of a semigroup of strongly positive contractions of $\mathcal{A}$.

1. Introduction. Let δ be a densely-defined $*$ -derivation on a C^* -algebra $\mathcal{A}$. One of the central problems in the theory of unbounded derivations is to determine whether a derivation is a generator, i.e., whether there exists a one-parameter group of $*$ -automorphisms $\{\beta_t: t \in \mathbf{R}\}$ of $\mathcal{A}$ such that for all $x \in D(\delta)$, the domain of δ , $\delta(x) = \lim_{t \rightarrow 0} (1/t)(\beta_t(x) - x)$, and such that $D(\delta)$ coincides with the set of all x for which the limit above exists. Recently a number of articles have appeared considering this situation for the case where δ commutes with a group $\{\alpha_g: g \in G\}$ of automorphisms of $\mathcal{A}$, i.e., $\alpha_g: D(\delta) \rightarrow D(\delta)$, all $g \in G$, and $\alpha_g \circ \delta = \delta \circ \alpha_g$. For example, Bratteli and Jørgensen have shown in [1] that a closed $*$ -derivation must be a generator if it commutes with a compact abelian group of automorphisms of $\mathcal{A}$ and annihilates the fixed point algebra $\mathcal{A}^\alpha$ of $\mathcal{A}$. Roughly speaking, their strategy is to decompose the algebra into spectral subspaces which are invariant under G (and also δ), to show that δ acts as a generator on each of these subspaces and to piece these results together to show that δ is a generator.

In this paper we consider an AF-algebra $\mathcal{A}$, $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$, a certain maximal abelian C^* -subalgebra $\mathcal{C}$ and a closed $*$ -derivation δ which annihilates $\mathcal{C}$. For $n \in \mathbf{N}$, we construct conditional expectations Φ_n , mapping $\mathcal{A}$ onto the C^* -algebra $\langle \mathcal{A}_n, \mathcal{C} \rangle$ generated by $\mathcal{C}$ and the finite-dimensional subalgebra $\mathcal{A}_n$ of $\mathcal{A}$. These maps are shown to respect the action of δ in the sense that $\Phi_n: D(\delta) \rightarrow D(\delta)$ and $\delta(\Phi_n(x)) = \Phi_n(\delta x)$, all $x \in D(\delta)$. Using these techniques, we show that δ is a generator, and that any two derivations satisfying the above conditions are strongly commuting. Moreover, we exhibit a dense $*$ -subalgebra of analytic elements of $\mathcal{A}$. We also show that δ is an approximately inner normal $*$ -derivation which is commutative in the sense of Sakai [9]. Finally, we show that our techniques may be applied to prove analogous results in the more general case where δ is a $*$ -semiderivation of $\mathcal{A}$.

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In a recent paper, [11], A. Kumjian uses somewhat different techniques to study similar questions on a class of C^* -algebras which includes the continuous trace AF-algebras. He shows that if δ is a closed $*$ -derivation on a continuous trace AF-algebra which annihilates the diagonal subalgebra, then δ must be a generator [11, Theorem 5.2]. Moreover, a dense subalgebra of analytic elements for δ is also exhibited.

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2. Diagonalization of an AF-algebra. In this section we introduce some notation which shall be needed in the proofs of our main results, and we also recall some facts and notation from [10] on the diagonalization of an AF-algebra. To begin, suppose that $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ is an ascending union of finite-dimensional C^* -algebras, with common identity $\mathbf{1}$. Let $\mathcal{A}$ be the unital AF-algebra formed as the uniform closure of the union $\bigcup_n \mathcal{A}_n$. For any subsets $S_1, S_2, \dots$ of $\mathcal{A}$, let $\langle \bigcup_n S_n \rangle$ denote the smallest C^* -subalgebra of $\mathcal{A}$ containing each of the S_n . In particular, $\mathcal{A} = \langle \bigcup_n \mathcal{A}_n \rangle$. Furthermore, if $\mathcal{B}$ is a C^* -subalgebra of $\mathcal{A}$, denote by $\mathcal{B}'$ the C^* -subalgebra of $\mathcal{A}$ given by $\{x \in \mathcal{A} : xy = yx, \text{ all } y \in \mathcal{B}\}$.

Following [10], we construct a maximal abelian C^* -subalgebra (m.a.s.a.) $\mathcal{C}$, the *diagonal subalgebra* of $\mathcal{A}$. $\mathcal{C}$ is defined inductively as follows: let $\mathcal{A}_0 = \mathbf{C} \cdot \mathbf{1}$, and for each $n \in \mathbf{N} \cup \{0\}$, choose a m.a.s.a. $\mathcal{D}_{n+1}$ of $\mathcal{A}'_n \cap \mathcal{A}_{n+1}$. Set $\mathcal{C}_0 = \mathbf{C} \cdot \mathbf{1}$ and define $\mathcal{C}_{n+1} = \langle \mathcal{C}_n, \mathcal{D}_{n+1} \rangle$. Then $\mathcal{C}_n$ is a m.a.s.a. of $\mathcal{A}_n$. Moreover, if I and J are index sets such that $\{p_i : i \in I\}$, $\{q_j : j \in J\}$ are the minimal projections of $\mathcal{C}_n$, $\mathcal{D}_{n+1}$, respectively, then $\{p_i q_j : i \in I, j \in J\}$ is the set of minimal projections of $\mathcal{C}_{n+1}$. A straightforward argument now shows that $\mathcal{C} = \langle \bigcup_n \mathcal{C}_n \rangle$ is a m.a.s.a. of $\mathcal{A}$. More generally, one also obtains that, for fixed n , and for any $k \in \mathbf{N}$, the abelian C^* -algebra $\mathcal{A}'_n \cap \mathcal{C}_{n+k}$ is a m.a.s.a. of $\mathcal{A}'_n \cap \mathcal{A}_{n+k}$, and the uniform closure $\langle \bigcup_k (\mathcal{A}'_n \cap \mathcal{C}_{n+k}) \rangle$ of the union is a m.a.s.a. of the AF-subalgebra $\mathcal{A}'_n$ of $\mathcal{A}$. We refer the reader to the exposition in [10] for details.

In [10] a conditional expectation Φ from $\mathcal{A}$ to its m.a.s.a. $\mathcal{C}$ is constructed. By conditional expectation it is meant that $\Phi : \mathcal{A} \rightarrow \mathcal{C}$ is a positive linear mapping which satisfies (i) $\Phi \circ \Phi = \Phi$, and (ii) $\Phi(xy) = \Phi(x)y$ and $\Phi(yx) = y\Phi(x)$ for $x \in \mathcal{A}$ and $y \in \mathcal{C}$. In particular it follows that $\Phi|_{\mathcal{C}}$ is the identity mapping. These properties determine Φ uniquely. We now derive a slight generalization of this notion.

PROPOSITION 2.1. *Let $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$ be a unital AF-algebra. Then for any $n \in \mathbf{N} \cup \{0\}$ there exists a conditional expectation Φ_n from $\mathcal{A}$ onto the C^* -subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_n$ and the (diagonal) m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$, i.e., $\Phi_n : \mathcal{A} \rightarrow \langle \mathcal{A}_n, \mathcal{C} \rangle$.*

Moreover, if $y \in \mathcal{A}'_n$, then $\Phi_n(y) \in \mathcal{A}'_n \cap \mathcal{C}$.

(REMARK. Observe that Φ_0 coincides with the conditional expectation Φ described above.)

PROOF. Fix positive integers $m > n$. We begin by defining a conditional expectation $\Phi_{m,n}$ from $\mathcal{A}$ onto $\langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$. First recall (see, e.g., [5]) that for any $x \in \mathcal{A}$ there exist elements $u_i \in \mathcal{A}_n$, $v_i \in (\mathcal{A}'_n \cap \mathcal{A}_m)$ and $w_i \in \mathcal{A}'_m$, $i = 1, 2, \dots, r$, such that $x = \sum_{i=1}^r u_i v_i w_i$. Now let $\{p_j : j \in J\}$, some index set J , be the minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_m$. Consider the linear map $\Phi_{m,n}$ defined by

$\Phi_{m,n}(x) = \sum_{j \in J} p_j x p_j$. Then

$$\begin{aligned} \Phi_{m,n}(x) &= \sum_{j \in J} \sum_{i=1}^r p_j u_i v_i w_i p_j = \sum_{j \in J} \sum_{i=1}^r u_i (p_j v_i p_j) w_i \\ &= \sum_{i=1}^r \sum_{j \in J} u_i (p_j v_i p_j) w_i = \sum_{i=1}^r u_i (\Phi_{m,n}(v_i)) w_i. \end{aligned}$$

Observe that for each i , $\Phi_{m,n}(v_i) \in \mathcal{A}'_n \cap \mathcal{A}_m$, and furthermore, $\Phi_{m,n}(v_i)$ commutes with $\mathcal{A}'_n \cap \mathcal{C}_m$. But $\mathcal{A}'_n \cap \mathcal{C}_m$ is maximal abelian in $\mathcal{A}'_n \cap \mathcal{C}_m$ [10], so that $\Phi_{m,n}(v_i) \in (\mathcal{A}'_n \cap \mathcal{C}_m)$. Hence $\Phi_{m,n}(x) \in \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$. Moreover, one checks easily that the latter algebra is fixed by $\Phi_{m,n}$, so that $\Phi_{m,n}$ is a conditional expectation onto $\langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$.

It is straightforward to verify that for $x \in \mathcal{A}_r$ and for $q > m \geq r$, $\Phi_{m,n}(x) = \Phi_{q,n}(x)$. To see this, note that since

$$\mathcal{A}'_n \cap \mathcal{C}_q = \mathcal{A}'_n \cap \langle \mathcal{C}_m, \mathcal{D}_{m+1}, \dots, \mathcal{D}_q \rangle = \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{D}_{m+1}, \dots, \mathcal{D}_q \rangle,$$

there exist minimal projections $e_1, \dots, e_s$ in $\langle \mathcal{D}_{m+1}, \dots, \mathcal{D}_q \rangle$ such that $\{p_j e_k : j \in J, 1 \leq k \leq s\}$ is the set of minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_q$. But since $e_k \in \mathcal{A}'_r$, all k , we have

$$\begin{aligned} \Phi_{q,n}(x) &= \sum_{j \in J} \sum_{k=1}^s p_j e_k x p_j e_k = \sum_{j \in J} \sum_{k=1}^s p_j x p_j e_k \\ &= \sum_{j \in J} p_j x p_j = \Phi_{m,n}(x). \end{aligned}$$

Similarly, one may show that for $q \geq m \geq n$, $\Phi_{m,n} \circ \Phi_{q,n} = \Phi_{q,n}$.

We now use the results above to establish that for all $y \in \mathcal{A}$, and for all fixed n , the sequence $\{\Phi_{m,n}(y) : m > n\}$ converges uniformly. For, if $y \in \mathcal{A}$, and $\varepsilon > 0$, there exists an index r and $x \in \mathcal{A}_r$ such that $\|y - x\| < \varepsilon/2$. Using $\|\Phi_{m,n}\| = 1$, all $m > n$, we have for $q \geq m \geq r$,

$$\begin{aligned} \|\Phi_{q,n}(y) - \Phi_{m,n}(y)\| &\leq \|\Phi_{q,n}(y - x)\| + \|\Phi_{q,n}(x) - \Phi_{m,n}(x)\| + \|\Phi_{m,n}(y - x)\| \\ &\leq \|y - x\| + 0 + \|y - x\| < \varepsilon. \end{aligned}$$

Hence the sequence $\{\Phi_{m,n}(y) : m > n\}$ has a uniform limit $\Phi_n(y)$. From the identity $\Phi_{m,n} \circ \Phi_{q,n} = \Phi_{q,n}$, $q \geq m \geq n$, it is straightforward to show that $\Phi_{m,n} \circ \Phi_n = \Phi_n$, and therefore, for all $y \in \mathcal{A}$,

$$\Phi_n(y) \in \bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle.$$

We show $\langle \mathcal{A}_n, \mathcal{C} \rangle = \bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$. First, we have, for all $m \geq n$ and $q \geq n$,

$$(\mathcal{A}'_n \cap \mathcal{C}_q) \subseteq (\mathcal{A}'_n \cap \mathcal{C}_m) \quad \text{if } m \geq q,$$

and

$$\begin{aligned} (\mathcal{A}'_n \cap \mathcal{C}_q) &= \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{D}_{m+1}, \dots, \mathcal{D}_{q+1} \rangle \\ &\subseteq \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle \quad \text{if } m < q, \end{aligned}$$

so that, in either case, we have $(\mathcal{A}'_n \cap \mathcal{C}_q) \subseteq \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$, and therefore,

$$\begin{aligned} \langle \mathcal{A}_n, \mathcal{C} \rangle &= \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C} \rangle = \left\langle \mathcal{A}_n, \bigcup_{q>n} (\mathcal{A}'_n \cap \mathcal{C}_q) \right\rangle \quad (\text{from [10]}) \\ &\subseteq \bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle. \end{aligned}$$

For the reverse inclusion, suppose $x \in \langle \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle$, all $m \geq n$. Then clearly $x \in \mathcal{A}'_n$ and $x \in (\mathcal{A}'_n \cap \mathcal{C}_m)'$, all m , hence $\mathcal{A}'_n \cap [(\mathcal{A}'_n \cap \mathcal{C})']$. But $\mathcal{A}'_n \cap \mathcal{C}$ is maximal abelian in $\mathcal{A}'_n$ [10, Proposition I.1.3], so that $x \in \mathcal{A}'_n \cap \mathcal{C}$. The equality

$$\bigcap_{m \geq n} \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C}_m, \mathcal{A}'_m \rangle = \langle \mathcal{A}_n, \mathcal{A}'_n \cap \mathcal{C} \rangle = \langle \mathcal{A}_n, \mathcal{C} \rangle$$

now follows immediately. Therefore Φ_n is a conditional expectation from $\mathcal{A}$ to $\langle \mathcal{A}_n, \mathcal{C} \rangle$, as asserted.

Finally, observe that if $y \in \mathcal{A}'_n$, then for $m > n$, $\Phi_{m,n}(y) = \sum_{j \in \mathcal{J}} p_j y p_j$ also lies in $\mathcal{A}'_n$. Since $\Phi_n(y) = \lim_m \Phi_{m,n}(y)$, the last statement of the proposition must hold. $\square$

3. Derivations annihilating $\mathcal{C}$. We now apply the preceding results to show that any closed $*$ -derivation δ on $\mathcal{A}$ which vanishes on the diagonal m.a.s.a. $\mathcal{C}$ must necessarily be a generator. Our techniques will enable us to construct explicitly a dense $*$ -subalgebra of analytic elements for $D(\delta)$ (Proposition 3.2). Moreover, a direct application of our results shows that the set of closed $*$ -derivations annihilating $\mathcal{C}$ forms a family of strongly commuting generators (Corollary to Theorem 3.3).

LEMMA 3.1. *Let δ be a closed $*$ -derivation vanishing on $\mathcal{C}$. For $n \in \mathbf{N} \cup \{0\}$, let Φ_n be the conditional expectation onto $\langle \mathcal{A}_n, \mathcal{C} \rangle$ constructed in §2. Then for any $x \in D(\delta)$, $\Phi_n(x)$ is also in $D(\delta)$ and $\delta(\Phi_n(x)) = \Phi_n(\delta x)$.*

PROOF. For $m > n$, recall that $\Phi_{m,n}(x) = \sum_{j \in \mathcal{J}} p_j x p_j$, where $\{p_j : j \in \mathcal{J}\}$ is the set of minimal projections of $\mathcal{A}'_n \cap \mathcal{C}_m$. Hence if $x \in D(\delta)$, then $\Phi_{m,n}(x) \in D(\delta)$, and

$$(1) \quad \delta(\Phi_{m,n}(x)) = \sum_{j \in \mathcal{J}} \delta(p_j x p_j) = \sum_{j \in \mathcal{J}} p_j (\delta x) p_j = \Phi_{m,n}(\delta x).$$

By Proposition 2.1, $\Phi_n(x)$ and $\Phi_n(\delta x)$ are the uniform limits of $\{\Phi_{m,n}(x) : m > n\}$ and $\{\Phi_{m,n}(\delta x) : m > n\}$. Combining this result with (1), we conclude from the closedness of δ that $\Phi_n(x) \in D(\delta)$ and that $\delta(\Phi_n(x)) = \Phi_n(\delta x)$. $\square$

PROPOSITION 3.2. *Let δ be a closed $*$ -derivation vanishing on $\mathcal{C}$. Then $D(\delta)$ contains a dense $*$ -subalgebra of analytic elements. In fact, if $x \in \bigcup_n \mathcal{A}_n$, then x is an analytic element for δ .*

PROOF. Fix $n \in \mathbf{N} \cup \{0\}$. Since $\mathcal{A}_n$ is finite-dimensional, it is isomorphic to a direct sum $\sum_{k=1}^{p_n} \mathcal{M}_{r_k}$ of $r_k \times r_k$ matrix algebras $\mathcal{M}_{r_k}$ over $\mathbf{C}$. Hence, for $1 \leq k \leq p_n$, one may choose matrix units $e_{ij}^k \in \mathcal{A}_n$ ($1 \leq k \leq p_n$, $1 \leq i, j \leq r_k$) satisfying the identities (i) $e_{ij}^k e_{pq}^k = \delta_{jp} e_{iq}^k$, and (ii) for $k \neq l$, $e_{ij}^k e_{pq}^l = 0$. Furthermore, we may assume that the matrix units have been chosen so that the diagonal elements e_{ii}^k lie

in $\mathcal{C}$. To show that $\mathcal{A}_n$ consists of analytic elements for δ it clearly suffices to show that the e_{ij}^k are analytic.

We begin by showing that each matrix unit e_{ij}^k lies in $D(\delta)$. For, if $1 > \varepsilon > 0$, there exists $x \in D(\delta)$ such that $\|x - e_{ij}^k\| < \varepsilon$. But then $e_{ii}^k x e_{jj}^k \in D(\delta)$ and we have

$$\|e_{ii}^k x e_{jj}^k - e_{ij}^k\| = \|e_{ii}^k (x - e_{ij}^k) e_{jj}^k\| \leq \|x - e_{ij}^k\| < \varepsilon.$$

Hence we may assume, without loss of generality, that $x = e_{ii}^k x e_{jj}^k$. Therefore, x admits a decomposition $x = e_{ij}^k y$, where $y \in \mathcal{A}'_n$ (see [5]); whence $\Phi_n(x) = \Phi_n(e_{ij}^k y) = e_{ij}^k \Phi_n(y)$. Let $d_1 = \Phi_n(y)$; then $d_1 \in (\mathcal{A}'_n \cap \mathcal{C})$, from Proposition 2.1, so that

$$\begin{aligned} \|e_{jj}^k d_1 - e_{ij}^k\| &= \|e_{ji}^k (e_{ij}^k d_1 - e_{ij}^k)\| \leq \|e_{ij}^k d_1 - e_{ij}^k\| \\ &= \|\Phi_n(x) - \Phi_n(e_{ij}^k)\| \leq \|x - e_{ij}^k\| < \varepsilon. \end{aligned}$$

Since $\varepsilon < 1$, the inequality above implies that d_1 is invertible in the commutative C^* -subalgebra $e_{jj}^k \mathcal{C}$, i.e., there exists $d_2 \in e_{jj}^k \mathcal{C}$ such that $d_1 d_2 = e_{jj}^k$. But $d_2 \in D(\delta)$, $\Phi_n(x) \in D(\delta)$, so that $e_{ij}^k = \Phi_n(x) d_2 \in D(\delta)$.

From Lemma 3.1,

$$\delta(\Phi_n(x)) = \delta(\Phi_n(e_{ii}^k x e_{jj}^k)) = \delta(e_{ii}^k \Phi_n(x) e_{jj}^k) = e_{ii}^k \Phi_n(\delta x) e_{jj}^k.$$

Hence $\delta(\Phi_n(x))$ has the form $\delta(\Phi_n(x)) = e_{ij}^k d_3$, $d_3 \in \mathcal{A}'_n \cap \mathcal{C}$. Then writing $d = d_3 d_2$ ($\in \mathcal{A}'_n \cap \mathcal{C}$),

$$\delta(e_{ij}^k) = \delta[\Phi_n(x) d_2] = [\delta(\Phi_n(x))] d_2 = e_{ij}^k d.$$

Iterating, we have, for any $r \in \mathbf{N}$, $\delta^r(e_{ij}^k) = e_{ij}^k (d^r)$, so that e_{ij}^k is easily seen to be an analytic element of $D(\delta)$. $\square$

REMARK. We note that a somewhat similar analysis to the foregoing is carried out in [8], where δ is a closed $*$ -derivation on a UHF algebra $\mathcal{A}$ of Glimm type n^∞ . There it is shown that if δ vanishes on the natural embedding of $S(\infty)$ into $\mathcal{A}$ ($S(\infty)$ is the group of finite permutations on countably many symbols) and satisfies $\tau \circ \delta = 0$, where τ is the unique trace on $\mathcal{A}$, then δ admits an extension to a generator on $\mathcal{A}$.

From the proof of the proposition it is clear that $\langle \mathcal{A}_n, \mathcal{C} \rangle \subseteq D(\delta)$, all $n \in \mathbf{N}$, and that $\delta: \langle \mathcal{A}_n, \mathcal{C} \rangle \rightarrow \langle \mathcal{A}_n, \mathcal{C} \rangle$. Indeed, for $1 \leq k \leq p_n$, let d_j^k , $1 \leq j \leq r_k$, be the diagonal elements satisfying $\delta(e_{1j}^k) = e_{1j}^k d_j^k$. Consider the (skew-hermitian) element

$$ih_n = \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} (e_{j1}^k) \delta(e_{1j}^k).$$

Then we have the well-known identity $\delta|_{\mathcal{A}_n} = \text{Ad}(ih_n)|_{\mathcal{A}_n}$, i.e., for $x \in \mathcal{A}_n$, $x = [ih_n, x] = x(ih_n) - (ih_n)x$ (see [5]). But

$$\begin{aligned} ih_n &= \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} (e_{j1}^k) \delta(e_{1j}^k) \\ &= \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} e_{j1}^k e_{1j}^k (d_j^k) = \sum_{k=1}^{p_n} \sum_{j=1}^{r_k} e_{jj}^k d_j^k, \end{aligned}$$

so that $h_n \in \mathcal{C}$. Extending to $\langle \mathcal{A}_n, \mathcal{C} \rangle$, it is straightforward to show that $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle} = \text{Ad}(ih_n)|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$ as well. From this identity it now follows easily that $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$ is a generator, with corresponding one-parameter group $\{\alpha_t^{(n)}: t \in \mathbf{R}\}$ given by

$$\alpha_t^{(n)}(x) = \exp(-ith_n)(x) \exp(ith_n), \quad x \in \langle \mathcal{A}_n, \mathcal{C} \rangle.$$

Applying [4, Theorem 3.2.51] one shows that the closure δ_0 of the derivation $\delta|_{[\cup_n \langle \mathcal{A}_n, \mathcal{C} \rangle]}$ is an (approximately inner) derivation on $\mathcal{A}$ with corresponding one-parameter group $\{\alpha_t: t \in \mathbf{R}\}$ satisfying

$$\alpha_t(x) = \lim_n \exp(-ith_n)(x) \exp(ith_n), \quad x \in \mathcal{A}.$$

We show that δ_0 coincides with δ . Clearly $D(\delta_0) \subseteq D(\delta)$. To verify the reverse inclusion let $x \in D(\delta)$. Then $x = \lim_n \Phi_n(x)$ and $\delta(x) = \lim_n \Phi_n(\delta x) = \lim_n \delta(\Phi_n(x))$. Since $\Phi_n(x) \in \langle \mathcal{A}_n, \mathcal{C} \rangle \subseteq D(\delta_0)$, the result holds. Hence we have the following.

THEOREM 3.3. *Let δ be a closed $*$ -derivation vanishing on the diagonal m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$. Then δ is a generator. In fact there exists a sequence $(h_n) \subseteq \mathcal{C}$ of hermitian operators such that δ generates the approximately inner dynamics $\{\alpha_t: t \in \mathbf{R}\}$, where $\alpha_t(x) = \lim_n \exp(-ith_n)(x) \exp(ith_n)$, $x \in \mathcal{A}$.*

REMARK. Since the sequence $(h_n)_{n \in \mathbf{N}}$ lies in $\mathcal{C}$, the hermitian operators are mutually commuting. Hence δ is a commutative $*$ -derivation in the sense of Sakai (see [9]).

We recall the following notion.

DEFINITION 3.1. Let δ, δ' be generators on a C^* -algebra $\mathcal{A}$. Then δ, δ' are said to be strongly commuting if their corresponding one-parameter groups $\{\alpha_t: t \in \mathbf{R}\}$, $\{\alpha'_t: t \in \mathbf{R}\}$ satisfy $\alpha_t \circ \alpha'_{t_1} = \alpha'_{t_1} \circ \alpha_t$, $t, t_1 \in \mathbf{R}$.

COROLLARY. *The set of generators which vanish on the diagonal m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$ form a family of strongly commuting derivations.*

PROOF. Let δ, δ' be two such derivations, and let $\{\alpha_t\}, \{\alpha'_t\}$ be their corresponding one-parameter groups. Then from the preceding theorem there exist operators $h_n, h'_n \in \mathcal{C}$ for $n \in \mathbf{N}$ such that

$$\alpha_t|_{\langle \mathcal{A}_n, \mathcal{C} \rangle} = \text{Ad}(\exp(-ith_n)), \quad \alpha'_t|_{\langle \mathcal{A}_n, \mathcal{C} \rangle} = \text{Ad}(\exp(-ith'_n)).$$

Since h_n, h'_n commute it is straightforward to verify that $\alpha_t(\alpha'_{t_1}(x)) = \alpha'_{t_1}(\alpha_t(x))$ for $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$ and $t, t_1 \in \mathbf{R}$. The result now follows by continuity. $\square$

4. Applications to semiderivations. Let $\mathcal{A}$ be a C^* -algebra. A linear operator $\delta: D(\delta) \rightarrow \mathcal{A}$ is said to be a *semiderivation* (or, alternatively, a *dissipation*) if it satisfies the following properties:

- (a) $D(\delta)$ is a uniformly dense $*$ -subalgebra of $\mathcal{A}$,
- (b) $\delta(x)^* = \delta(x^*)$, all $x \in D(\delta)$, and
- (c) $\delta(x^*x) \geq \delta(x^*)x + x^*(\delta x)$ for $x \in D(\delta)$.

A central problem in the theory of semiderivations is to determine when δ is the generator of a strongly continuous one-parameter contraction semigroup $\{\alpha_t: t \in \mathbf{R}_+\}$ of positivity-preserving maps. For semiderivations vanishing on $\mathcal{C}$ we have the following analogues to Theorem 3.3 and its corollary. (We thank P. E. T. Jørgensen for suggesting this extension of our original results.)

THEOREM 4.1. *Let δ be a closed semiderivation annihilating the m.a.s.a. $\mathcal{C}$ of $\mathcal{A}$. Then δ is the generator of a strongly continuous one-parameter contraction semigroup $\{\alpha_t: t \in \mathbf{R}_+\}$ of symmetric, strongly positive maps, i.e., $\alpha_t(x^*) = \alpha_t(x)^*$ and $\alpha_t(x^*x) \geq \alpha_t(x)^*\alpha_t(x)$ for all $x \in \mathcal{A}$, $t \in \mathbf{R}_+$.*

PROOF. We preserve the notation of Proposition 3.2. Since $\mathcal{C} \subseteq D(\delta)$ and $\delta|_{\mathcal{C}} \equiv 0$, we have, by [3, Lemma 1.1],

$$(2) \quad \delta(xy) = (\delta x)y, \quad \delta(yx) = y(\delta x), \quad y \in \mathcal{C}, \quad x \in D(\delta).$$

Now using (2) we may employ the “averaging” argument of Lemma 3.1 to conclude that for all $n \in \mathbf{N} \cup \{0\}$ and $x \in D(\delta)$, $\Phi_n(x) \in D(\delta)$ and $\delta[\Phi_n(x)] = \Phi_n(\delta x)$. This implies that $\langle \mathcal{A}_n, \mathcal{C} \rangle \subseteq D(\delta)$ for all n , and

$$\delta: \langle \mathcal{A}_n, \mathcal{C} \rangle \rightarrow \langle \mathcal{A}_n, \mathcal{C} \rangle.$$

We consider $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$. Since δ is everywhere defined on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ it follows from [6, Theorem 1] that $\delta|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}$ is both dissipative and bounded. In particular, $\langle \mathcal{A}_n, \mathcal{C} \rangle$ consists of analytic elements for δ . Now, let $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then x may be decomposed as

$$x = \sum_{k=1}^{p_n} \sum_{i,j=1}^{r_k} e_{ij}^k c_{ij}^k, \quad c_{ij}^k \in \mathcal{C}.$$

Following the proof of Proposition 3.2, there exists for each matrix unit $e_{ij}^k \in \mathcal{A}_n$ an element $d_{ij}^k \in \mathcal{C}$ such that $\delta(e_{ij}^k) = e_{ij}^k d_{ij}^k$. Then using (2) repeatedly we have, for $c_{ij}^k \in \mathcal{C}$ and $r \in \mathbf{N}$,

$$(3) \quad \delta^r(x) = \sum_{k=1}^{p_n} \sum_{i,j=1}^{r_k} (e_{ij}^k c_{ij}^k) (d_{ij}^k)^r.$$

Define δ_0 to be the semiderivation $\delta_0 = \delta|_{D(\delta_0)}$, where $D(\delta_0) = \bigcup_n \mathcal{A}_n$. Then δ_0 is a dissipative semiderivation whose domain contains a dense $*$ -subalgebra of analytic elements. Hence δ_0 is closable, and its closure $\overline{\delta_0}$ is a generator of a contractive semigroup $\{\alpha_t: t \in \mathbf{R}_+\}$ [2, Theorem 5]. Now applying an argument identical to the one in the last paragraph of the proof of Theorem 3.3, we conclude that $\overline{\delta_0}$ coincides with δ .

We now consider the semigroup $\{\alpha_t\}$. Since $\delta(x)^* = \delta(x^*)$ for $x \in D(\delta)$, it follows immediately that $\alpha_t(x^*) = \alpha_t(x)^*$, all $x \in \mathcal{A}$. We show that the maps α_t are strongly positive. For $n \in \mathbf{N}$, consider any $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then since δ is a bounded dissipative semiderivation on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ with $\delta(1) = 0$, it follows from [7, Corollary 3] that $\{\alpha_t|_{\langle \mathcal{A}_n, \mathcal{C} \rangle}: t \in \mathbf{R}_+\}$ is strongly continuous and that $\alpha_t(x^*)\alpha_t(x) \leq \alpha_t(x^*x)$. Using the strong continuity on $\langle \mathcal{A}_n, \mathcal{C} \rangle$ for each $n \in \mathbf{N}$, it follows immediately that the semigroup of contractions is strongly continuous on $\mathcal{A}$. Now, using continuity, it is straightforward to show that $\alpha_t(x^*)\alpha_t(x) \leq \alpha_t(x^*x)$, all $x \in \mathcal{A}$, so that the semigroup is strongly positive. $\square$

COROLLARY. *Let δ, δ' be any two semiderivations on $\mathcal{A}$ which annihilate $\mathcal{C}$. Then δ and δ' are strongly commuting generators of contraction semigroups.*

PROOF. From the theorem we deduce that δ (respectively, δ') are the generators of the strongly continuous contractive semigroups $\{\alpha_t: t \in \mathbf{R}_+\}$ (respectively,

$\{\alpha'_t: t \in \mathbf{R}_+\}$. Fix $n \in \mathbf{N}$, and consider $x \in \langle \mathcal{A}_n, \mathcal{C} \rangle$. Then, by the theorem, x is an analytic element for both δ and δ' , i.e.,

$$\alpha_t(x) = \sum_{n=0}^{\infty} [(t\delta)^n(x)]/(n!) \quad \left(\text{respectively, } \alpha'_t(x) = \sum_{n=0}^{\infty} [(t\delta')^n(x)]/(n!) \right).$$

Using the formula in (3) for $\delta^r(x)$, $r \in \mathbf{N}$ (and a similar formula for $(\delta')^s(x)$, $s \in \mathbf{N}$), it is trivial to verify that $[\delta^r \circ (\delta')^s](x) = [(\delta')^s \circ \delta^r](x)$. But from this identity it follows immediately that for $t, t_1 \in \mathbf{R}$, $\alpha_t(\alpha'_{t_1}(x)) = \alpha'_{t_1}(\alpha_t(x))$. By continuity this equation holds for all $x \in \mathcal{A}$, so that δ, δ' are strongly commuting semiderivations. $\square$

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DEPARTMENT OF MATHEMATICS, UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MARYLAND 21402